The Index Theorem

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Preface

The goal of this essay is to give most of the prerequisite knowledge (modulo various technical lemmas and theorems required to prove the main statements) in order to understand the index theorem stated in [1] which we will finally state at the end of this essay; in doing so I will assume some basic knowledge that one might expect from having taken courses on differential geometry,analysis, and algebra (i.e., Mat367,Mat357, Mat347 at UofT) and some basic comprehension of category theory. The main topics I have chosen to focus on are: Elliptic Operators, Fredholm Operators, Topological K-theory (and subsequent variations of it), Topological Index, Analytic Index, and finally the statement of the index theorem by uniqueness.

1 Elliptic Operators

An important class of operators we will study are elliptic operators, as later we will be focused on the analytic index arising from their study, and so we describe them here first. In writing this section I found [2] a useful resource.

Differential Operator of order r First let us define a differential operator of order r on an n dimensional manifold (written in coordinates on an open set $U \subset \mathbb{R}^n$). In what follows we let $D_i = \frac{d}{dx_i}$, i.e., the coordinate derivative, and $D_i^{\alpha_i}$ is just D_i applied α_i times. To generalize notation we will then define $\alpha = (\alpha_1, \cdots, \alpha_n)$, and then $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$. Then an operator of order r is of the form $P = \sum_{|a| \leq r} f_\alpha(x) D^\alpha$ where $f_\alpha \in C^\infty(U)$. Note that this operator is a map from from $C^\infty(U) \to C^\infty(U)$

Symbol of an operator A then useful characterization of a differential operator is its symbol $p(x, \epsilon) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, which is defined as $p(x, \varepsilon) = \sum_{|\alpha|=r} f_\alpha(x) \varepsilon^\alpha$ where it is understood that if $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$ then $\varepsilon^\alpha = \sum_i^n \varepsilon_i^{\alpha_i}$ (i.e. a type of polynomial).

Defining Elliptic Operators In general we will identify the contangent space $T_x^* M$ of the manifold M at x with $\mathbb{R}^n$. We then say a differential operator P is *elliptic* if $p(x,\varepsilon) \neq 0$ for all $x \in M$ and $\varepsilon \in T_x^*M - 0$ (where one just uses the coordinate deffinition of p given earlier).

With this definition, a simple example of an elliptic operator is given by $P = \sum D_i^2$, i.e., the laplacian. A general rule of thumb is that one wants the highest order terms to be such that the polynomial created by replacing D_i with x_i is only 0 at the origin, hence another example is just $P = D_1$ corresponding to a linear line (note this rule of thumb applies for constant coefficients).

When we also are working over the Sobolev Space of $C^\infty(M) \cap L^2(M)$, which has a natural inner-product, we can in fact discuss the transpose of an elliptic operator (which is still an elliptic operator) defined as the unique operator such that $\langle Pu, v\rangle = \langle u, P^T, v\rangle$. We can in fact give a local definition of P^T by taking if the volume element $dx = wdx_1 \cdots dx_n$ then $P^T = \frac{1}{w}\sum_{|\alpha|\leq r} D^\alpha \bar{a^\alpha} w$, where $P = \sum_{|\alpha|\leq r} a^\alpha D^\alpha$. Note this formula comes simply from the definition of the inner product of $L^2(M)$

2 Fredholm Operators

We now move onto fredholm operators, and the notion of fredholm index which is fundamentally what underlies the notion of *analytic index* described later Section 6. In writing this section I found [3] a helpful resource.

Fredholm Operator Let X, Y be banach spaces, and let $T : X \to Y$ be a bounded linear operator. If moreover $Ker(T), CoKer(T)$ are finite dimensional and $Ran(T)$ is closed (this last statement is redundant and follows from $CoKer(T)$ being closed), we then call T a *Fredholm Operator.*

In fact we have an explicit example of Fredholm Operators in elliptic operators over compact manifolds. This is stated as a theorem

Theorem 1. *If manifold X is compact, and P is an elliptic differential operator on it, then $Ker(P)$ is finite, $CoKer(P) = Ker(P^T)$ which is also finite dimensional (as P^T is elliptic).*

Proof. refer to the text on elliptic operators [2] □

Fredholm Index To a Fredholm operator we can define an index by $ind(T) = dim(Ker(T)) - dim(CoKer(T))$. From this definition we can also simply state that a bounded linear operator T is Fredholm iff its index is well-defined and finite: the forward direction of this statement is by definition, and the reverse is simply the conditions for both $Ker(T)$ and $CoKer(T)$ to be finite (noting the well-defined removed the case of taking $\infty - \infty$). Regardless, what this gives is a natural analytic (or algebraic) type index associated to elliptic operators which we extend in Section 6.

3 Defining Topological K Theory

The goal of this essay is to state the Atiyah-Singer theorem as it relates k-theory to fredholm index, but to do that we must first venture into topoligcal k-theory. An alternative formulation can be found with cohomology, but as Atiyah said, k-theory is far more natural. In what follows I first describe vector bundles, which I found [4] to be a good reference on it, and then subsequently go to k-theory where I found [5] to be a good reference.

Defining Vector Bundles One can think of a vector bundle as a juxtaposition of a vector-space onto every point of a manifold (and in fact this juxtaposition creates an entirely new larger manifold). An example is simply the tangent space TpM on M which gives a smooth manifold structure $M \times \mathbb{R}^n$. More formally we define a *Vector Bundle of* M *of rank* k as a map $\pi : E \to M$ where M is an n-dim manifold, and E is an $n + k$-dim manifold, that satisfies the following:

1. $E_p = \pi^{-1}(p)$ has a real k-dim vector space structure for every $p \in M$ (this is arguably why it is called a "vector" bundle)
2. We have a local trivializing cover of M, i.e., an open cover $\{U_i\}$ of M, such that there is diffeomorphism $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ and a map β such that $\beta \circ \phi_i = \pi|_{\pi^{-1}(U_i)}$ (one can think of this as the map π factoring through a trivial vector bundle locally)
3. Furthermore $\phi_i|_{E_p} : E_p \to p \times \mathbb{R}^n$ is a vector space isomorphism for all i

Note that given such a vector bundle, we will often refer to it as a rank k vector bundle with basis M

Equivalence Classes of Vector Bundles The focus here is to develop a type of algebraic geometry built on studying the vector bundles a manifold accepts; yet as always we'd like to modulo out suitably "equivalent vector bundles". If $\pi_0 : E_0 \to M$ and $\pi_1 : E_1 \to M$ are two rank k vector bundles with basis M, we say they are *isomorphic* if there exists a diffeomorphism $\phi : E_0 \to E_1$ such that π_0 factors through π_1 with ϕ, i.e $\pi_0 = \pi_1 \circ \phi$, and that $\phi|_{\pi_0^{-1}(p)} : \pi_0^{-1}(p) \to \pi_1^{-1}(p)$ is a vector space isomorphism for all $p \in M$. That is not only are E_0 and E_1 diffeormorphic as manifolds, but they are in a way that does not change their vector bundle structure (upto isomorphism).

Giving Vector Bundles a Ring Structure We begin by first noting that we naturally have two operations on vector bundles of fixed rank k that one might consider addition and multiplication. Given two vector bundles E_0, E_1 of dimension l, m with basis M (note dimension is different than rank), we have $E_0 \oplus E_1$ is a dimension $l + m$ vector bundle by the map $\pi_0 \oplus \pi_1$ with the inverse $(\pi_0 \oplus \pi_1)^{-1}(p) = \pi_0^{-1}(p) \oplus \pi_1^{-1}(p)$ (i.e., $E_0 \oplus E_1$ is the union of the direct sum of fibers $\cup_{p\in M}(\pi_0 \oplus \pi_1)^{-1}(p)$ which inherits a smooth manifold structure and a vector bundle structure). We then also define a "mulitplication" by $E_0 \otimes E_1$, which is an lm-dim vector bundle given by the map $(\pi_0 \otimes \pi_1)^{-1}(p)$ defined by $(\pi_0 \otimes \pi_1)^{-1}(p) = \pi_0^{-1}(p) \otimes \pi_1^{-1}(p)$ (i.e., $E_0 \otimes E_1$ is the union of the tensor product of fibers $\cup_{p\in M}(\pi_0 \otimes \pi_1)^{-1}(p)$ which inherits a smooth manifold structure and a vector bundle structure). Note both of these operations are commutative after modulo the equivalence of vector bundles (as then swapping order doesn't do anything).

Thus, with those two operations we have the equivalence classes of vector bundles on M, $Vect(M)$, is now a semi-ring; recall a semi-ring is a ring without the condition of additive inverses. However we can artificially make it have a ring structure by the grothendick construction over the semi-group given by the "addition" $\oplus$ operator.

Theorem 2 (Grothendick Ring)**.** *For any semi-ring A, there is a ring $\mathfrak{A}$ with homomorphism $g : A \to \mathfrak{A}$ such that any homomorphism from A to a ring R factors uniquely through $\mathfrak{A}$ and g*

Proof. We'll just state the construction of $\mathfrak{A}$ which is simply $A \times A/$ where (a, b) (c, d) is exists $z \in A$ such that $a + d + z = b + c + z$. Then we define $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)\dot{(}c, d) = (ac + bd, ad + bc)$; note this is a ring as we now have the additive inverse $-(a, b) = (b, a)$ as then $(0, 0)$ $(a, b) - (a, b) = (a + b, a + b)$. We refer the reader to "An Introduction to K-theory" by M Rordam for the rest of the proof, and more properties of the grothendick construction. □

With this we then define $g(Vect(M))$ the *virtual bundles* over M

Defining the K theory We will call the grothendick construction on the isomoprhic classes of complex vector bundles on the manifold M (where we have the same definition as before, but now that they are complex vector spaces locally) the *K-theory of M.* In particular our map $M \to K(M)$ is a contravariant functor from topoligcal spaces (as manifolds are also topological spaces) to rings.

K theory of one point sets As a useful example, note that if $X = \{x\}$, i.e. one point set, then $K(X) = \mathbb{Z}$. This is as every vector bundle is simply a vector space, and they are all trivially vector bundle isomorphic upto dimension, and thus our semi-ring of equivalence classes of vector bundles are isomorphic to the semi-ring $\mathbb{N}$ (as the map $[E] \to dim(E)$ is a ring isomorphism under the operations we defined earlier, where our vector bundle sum in facts sums dimension, and our vector bundle multiplication in fact multiplies dimension). Then applying the grothendick construction to get $K(X)$, and noting the grothendick construction on $\mathbb{N}$ gives $\mathbb{Z}$, we have $K(X) \cong \mathbb{Z}$

4 Building on K-theory

We now go through some of the basic properties of the k functor.

A natural K theory homomorphism If given a map between manifolds $f : X \to Y$ and a vector bundle $\pi : E \to Y$ on Y we can define the pullback q of π by f as $f^*E = X \times_Y E = \{(x, v) \in X \times E : \pi(v) = f(x)\},\ q(x, v) = x$ and note this will define a vector bundle over X. It's then easy to see this then gives a homormorphism between vector bundles of X and Y (under the semi-group operations we defined, noting the important fact was how they're note defined as regular direct sums and tensor products, but the union of them done on fibers), and so moreover $f^* : K(Y) \to K(X)$ is a ring homomorphism. We in fact have this homomorphism is invariant over homotopy class.

Theorem 3. *Given $f_0, f_1 : X \to Y$ that are homotopic, and $\pi : E \to Y$ is a vector bundle over Y, we have $f_0^*(E) = f_1^*(E)$ (upto equivalence classes)*

Proof. we refer the reader to the text on topological k-theory □

Reduced K Theory Recall the example given in the earleir section where $K(\{x\}) \cong \mathbb{Z}$; in some way this is a part of every pointed space, and we can formalize this as follows. Note that the embedding $i : \{x\} \to X$ for some point $x \in X$ induces a surjective ring homomorphism $i^* : K(X) \to K(\{x\}) \cong \mathbb{Z}$. Then by the isomorphism theorem we have $K(x) \cong Ker(i^*) + \mathbb{Z}$. In particular we see the $\mathbb{Z}$ component is a part of every pointed K-theory, so without much loss of generality we can simply forget this part and focus on the $Ker(i^*)$ which we call *the reduced k-theory* and denote as $\tilde{K}(x)$

Bott Periodicity One of the fundamental results in k-theory is a natural 2-periodicity when applying a certain operation called the reduced suspension ΣX to a space X. The suspension SX of a space X is simply $X \times [0, 1]/\Gamma where$ indentifies $X \times 0$ as a point and $X \times 1$ as another point. We then define the reduced suspension ΣX of a pointed space X by further identifying the line $\{x_0\} \times [0, 1]$ to a single point, i.e., $\Sigma X = X \times [0, 1]/((X \times 0) \cup (X \times 1) \cup (\{x_0\} \times [0, 1]))$. Then Bott preiodicity gives the following

Theorem 4 (Bott Periodicity)**.** *$\tilde{K}(X) \cong \tilde{K}(\Sigma^2 X)$, where $\Sigma^2 X$ is the reduced suspension of ΣX.*

Proof. We refer the reader to a text on k-theory for the proof, but one approach is by using facts about $K(S^2)$ □

K_G-Theory There is a generalization of the spaces we have been considering so far (smooth manifolds) by allowing them to admit an action by a compact lie group G; we call such a space with an action by G a G-space. Additionally we call a G-vector bundle a regular vector bundle of X but which now has a group action G on the fibers E_x such that $g : E_x \to E_{g(x)}$ is linear. Thus now restricting to only G-vector bundles over a G-space X we analagoulsy (defining the same operations for a semi-ring and extending to a ring by the grothendick construction) form a ring $K_G(X)$.

Remark on K_G The important reason for why we'll consider K_G is that we'll define index as a map from $K_G(TX) \to R(G)$, where $R(G)$ is the *representation ring*. Formally $R(G)$ is a ring defined by the finite dimensional representations (upto isormorphism equivalence classes) of G over the field $\mathbb{C}$ after the grothendick construction; elementary representation theory tells one that direct sums and tensor products of representations are well-defined and so the finite dimensional representation are a semi-ring, and thus applying the grothendick construction we obtain our ring $R(G)$.

With that said, we then naturally get that $K_G(\{x\}) = R(G)$, which follows from the fact that the isomorphism classes of vector bundles of the one point set simply define finite-dimensional representation of G (upto isomorphism), as a finite dimensional representation of G is simply $\pi : G \to GL_n(\mathbb{C})$ (and we now have the G-vector bundles of $\{x\}$ of rank k are K-dimensional representations of G). So from $Vect(\{x\}) \cong$ semiring of finite representations of G, by the grothendick construction on both sides we get $K(\{x\}) \cong R(G)$

5 Topological Index

The goal in this section will be to define (or perhaps more correctly, describe) the map t-ind : $K_G(TX) \to R(G)$ which is called the *topological index*. In particular we will take X to also be compact, and proceed by first considering $X \subset Y$ for some G-space Y.

Defining $i_! : K(TX) \to K(TY)$: The process is as follows, one can first consider a tubular neighbourhood N of X in Y; this then gives a tubular neighbourhood TN of TX in TY (note that formally TN is in fact a complex vector bundle over TX); we then use the *thom homomorphism* defined earlier in Atiayh and Singer's paper [1] (pg. 11) to get a homomorphism $\phi : K(TX) \to K(TN)$. Then using the simple inclusion $k : TN \to TY$ and then the pushforward $k_* : K(TN) \to K(TY)$, we define a map $i_! = k_* \circ \phi$.

Getting t-ind Now keeping X as a compact differentiable G-manifold, we have the existence of real representation space E of G such that we have a differential embedding $i : X \to G$; as noted by the authors, this follows from Peter Weyl Theorem with a proof found in [6]. Furthermore we also have the inclusion $j : P = \{0\} \to E$ and thus from the above we have the maps $i_! : K_G(TX) \to K_G(TE)$ and $j_! : K_G(TP) \to K_G(TE)$. However in fact $j_!$ is actually just the Thom Homomorphism (as the tubular neighbourhood TN of TP is TE), and in fact an earlier statement in the paper [1] (pg. 12) showed that Thom Homomorhpism in this case is an isomorphism. Lastly recall from the remark in our earlier section defining K_G that $K_G(TP) \cong R(G)$ as P is a one point set.

Thus we obtain t-ind by defining t-ind $= i_! \circ j_!^{-1}$. This defintion satisfies various properties, such as being well-defined (i.e., independent of the embedding i) but we will omit these details in our discussion and refer the reader to the original paper.

6 Analytic Index from K-theroy

As defined already in Section 2 we already have a notion of index for an elliptic operator P, however we will like to extend the tool box we have have to describe this index to a map a-ind : $K_G(TX) \to R(G)$ analogously to how t-ind is defined; beyond the aesthetic nicety of having both indexes defined similarly, we will see later that this formulation is in fact the main idea behind the index theorem.

Some more facts about Fredholm Index We will state these extra details as lemmas here, and leave the proofs for the reader to find in their desired textbook on Fredholm Operators (notably the latter 2 are more or less trivial). These are stated as properties 6.2 - 6.5 in the index paper [1].

Lemma 1 (Homotopy Equivalence of Index)**.** *(Fredholm) Index is invariant on the homotopy class of symbols (in the space of invertible symbols)*

Lemma 2 (Symbol Equivalence to Index Equivalent)**.** *If the symbol of two elliptic operators P and Q are equal for all ε in the unit ball of TX, for some metric, then $ind(P) = ind(Q)$*

Lemma 3. $ind(P \oplus Q) = ind(P) + ind(Q)$

Lemma 4. $ind(P) = 0$ *means restricted to a certain vector bundles E, F in the range and domain respectively, P is an isomorphism from E to F.*

Analytic Index a-ind Now the first thing to note is that as defined, the symbol $p(x, \varepsilon)$ is in fact a vector bundle of TX, and hence is an element of $K(TX)$. Furthermore by the previous lemmas (and facts about homogenous complexes of length one which we have not described here) one gets the index of P depends only on the class of $p(x, \varepsilon) \in K(TX)$, and furthermore that there is a homomorphism a-ind : $K(TX) \to \mathbb{Z}$ induced by the Fredholm index.

Generalized to K_G: We now consider G-invariant elliptic operators P (when applied to x and ε variables). We will define $ind(P) = [ker(P)] - [CoKer(P)] \in R(G)$ which note makes sense as $Ker(P)$ and $CoKer(P)$ are both invariant under G actions and so are (given the algebra structure inherited from $K(TX)$) G-modules, and moreover finite dimensional by the fact our regular index exists. Furthermore it can be shown that this still satisfies the necessary properties of index that made $ind(P)$ only dependent on $K_G(TX)$ class of its symbol (one simply reproduces the above lemmas stated earlier, or the properties needed for them, which Atiyah and Singer did in the paper [1]).

Hence more generally we define a-ind : $K_G(TX) \to R(G)$ as induced by the map $ind(P) = [Ker(P)] - [CoKer(P)] \in R(G)$.

7 The Index Theorem

As was suggested in our opening discussion in Section 6, the main idea behind index theorem, and how it generalizes to other notions of index, is by standardizing the form of what an index map is. This is presented as the index function (a functorial homomorphism) and the two axioms of index functions.

The Index Function Consider an arbitrary $R(G)$ homomorphism ind_G^X : $K_G(TX) \to R(G)$ such that it is functorial with respect to diffeomorphisms of X and homomorphism of G. That is if f is a diffeomorphism between X and Y then $ind_G^X = ind_G^Y \circ f^*$, and if $\phi : G' \to G$ is a group homomorphism then $ind_{G'}^X \circ phi^* = phi \circ ind_G^X$. Such a map is called an *index function*.

The Uniqueness of Index There are then two further important axioms which then give us the desired characterization (or uniqueness) of index functions *ind*, stated as:

1. A1: If X is a point, then *ind* is an identity map from $K_G(TX) \cong R(G) \to R(G)$
2. A2: *ind* commutes with the $i_!$ map given in Section 5

With these axioms we then have one of the major results of the paper:

Theorem 5 (Uniqueness of Index Functions)**.** *Any index function ind which satisfies the two axioms is then equal to t-ind*

Proof. We refer the reader to the paper [1] as it follows rather trivially from various definitions (though involves several diagrams) □

In practice it is noted that verifying A2 is difficult, and so Atiyah and Singer introduce several more elementary axioms which they then show implies A2; nevertheless the above statement is the crux of the "uniqueness" of index, and ultimately the index theorem.

Applying to a-ind Finally the second half of the paper [1] proves that the analytic index, given by the a-ind map we stated in 6, is in fact an index function as defined above. Moreover, Atiyah and Singer show it satisfies both A1 and A2. Thus we are left with the final remarkable result (which one might call the index theorem)

Theorem 6 (Index Theorem)**.** *a-ind = t-ind*

References

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