Irrational Rotational Algebras

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1 introduction

In this essay we survey different aspects of the irrational rotational algebra, and attempt to consolidate all the major work into one text. In doing so we omit proofs (the reader is pointed to the references), however I do elaborate on them, and in some cases fill in technical details that were left from the proof in the paper but were perhaps not completely obvious (to me).

2 Definition of A_α

The first definition is based on how the algebra was defined in [8], where as the second is the alternative definition seen in [6] and [7] and our textbook; they are in fact the same, however I felt the way they're introduced are sufficiently different that they give different views on the same algebra.

The third definition isn't used explicitly in the work I reference later on, but is mentioned as an aside in exercise 5.8 of [9], and it doesn't hurt to mention here (and helps explain why sometimes the algebra is referred to as a crossed product).

2.1 Rieffel Definition

Consider the unit circle in the complex plane $\mathbf{T}$ (or real plane, however they are really just the same, and the complex plane lends it's self to easier construction and classification of properties). A natural action on the circle is the rotation by some angle $\alpha\ in[0, 2\pi)$, but it's rather trivial to see that repeated rotations by $\alpha \in Q$ create finite cyclic groups, which are rather well understood. What's perhaps less intuitively understood is the behaviour of rotations by irrational α, which aren't necessarily finite. For simplicity, from now on we let $\alpha \in (0, 1)\ Q$, and when I refer to a rotation by α I refer to $2\pi\alpha$.

In order to characterize these actions, let us first focus on the set of L^2 complex-valued integrable functions on $\mathbf{T}$ denoted $L^2(\mathbf{T})$. We can then characterize the rotations by an irrational α as a unitary operator S_α on $L^2(\mathbf{T})$, and can consider the family of rotations by α as the C* algebra generated by S_α with pointwise multiplicative operators M_f, denoted $C*(S_\alpha)$ or from now on A_α; this is the irrational rotation algebra. When it is understood or irrelevant in context, I will simply write S instead of S_α

2.2 Unitary Characterization

We can also characterize A_α as the algebra generated by a pair of unitary operators u, v such that $u^*vu = e^{2\pi i\alpha}v$; an explicit form for these are given in exercise 5.8, however it's worth noting there they consider operators on $L^2(\mathbf{T}^2)$ where as [8] considers operators on $L^2(\mathbf{T})$. In general A_α is isomorphic to any algebra generated by any unitaries U, V such that $VU = \lambda UV$ where $\lambda = e^{2\pi\alpha}$; this is the universal property of A_α.

The introduction of [7] also gives another intuitive characterization of u, v in A_α as the rotation by $2\pi\alpha$ and the "identity" imbedding into the complex unit circle respectively.

2.3 Crossed Product

Let $\phi(z) = wz$ which maps $\mathbf{T}$ to $\mathbf{T}$, where $w = e^{2\pi * i\alpha}$. We can then also define A_α as the crossed product $C(\mathbf{T}) \times_\theta Z$, where θ is an automorphism of $C(\mathbf{T})$ given by the pull back with ϕ.

Crossed products have their own universal property which lead to characterizing the above universal property, and various other facts lead to the simplicity of A_α, and the unique tracial state. The universal property of the crossed product is defined in terms of a bijection between unital representations and unital *-homomorphisms, and I refer the reader to online resources to read more on them; I found "Lecture 2: Introduction to Crossed Products and More Examples of Actions" by the university of Oregon informative.

3 Traces

The following are handy facts related to the trace of A_α.

3.1 explicit formulations

As is defined in exercise 5.8 in the textbook, we have a tracial state on A_α given by $t(a) = \langle aE_0, E_0\rangle$, where E_0 is the function in $L^2(\mathbf{T})$ that maps to one. As α is irrational, we have that this is in fact the unique tracial state on A_α.

For a dense subset of A_α of the form $\sum M_{f_n} S^n$, we have by [8] that the trace is:

$$t(\sum M_{f_n} S^n) = \int_{\mathbf{T}} f_0(t)dt \tag{1}$$

where this is using the normalized Lebesgue measure for (T), i.e $\int_0^{2\pi} f_0(e^{i*t})dt$.

By exercise 5.8, we also have the following characterization of the trace when $p \in A_\alpha$ is of the form $p = f(u)v^* + g(u) + vf(u)$, where f, g are continuous functions from $\mathbf{T}$ to R:

$$t(p) = \int_{\mathbf{T}} g(z)dz \tag{2}$$

where dz is the normalized haar measure on $\mathbf{T}$.

3.2 Properties

From the theorem in [6] (see discussion of K_0 groups) and theorem 1 in [8] (see discussion on projections and equivalence), we have the following statement, given as a corollary in [6]

Corollary (Range of trace). *Let ϕ be the homomorphism from $K_0(A_\alpha)$ to R given by the trace t, then we have $phi(K_0(A_\alpha)) = Z + Z\alpha$*

4 Projections and Equivalence Conditions

What I'll proceed to discuss/survey in this section are the main results of [8], and highlight certain technical details I found interesting.

4.1 Theorem 1

The statement of the theorem is:

Theorem (Theorem 1). *For every $\beta \in (Z + Z\alpha) \cap [0, 1]$ there is a projection p in A_α s.t $t(p) = \beta$*

The first thing to note is that in theorem 1.1 in the paper, we have the more explicit formulation, where p is specifically stated to be supported on $-1, 0, 1$, which means it is of the form:

$$p = M_h S^-1 + M_f + M_g S \tag{3}$$

i.e consists of powers of S in $-1, 0, 1$. This is in fact analogous to the formulation of p in exercise 5.8, and we in fact see question vii) gives the three conditions used in the proof of theorem 1 (the entire problem is in fact very reminiscent of the proof for this theorem).

Now the work done by [6] gives that the range of the trace is contained in $Z + Z\alpha \cap [0, 1]$ (look at discussion of k_0 group for more details) and so directly by theorem 1.1 you get theorem 1.2.

4.2 Theorem 2

The statement of the theorem is:

Theorem (Theorem 2). *If α and β are irrational number both in $[0, 1/2]$, and A_α and A_β are isomorphic, then $\alpha = \beta$. If α is any irrational number, with fractional part α, let $\beta = 1 - \alpha or \alpha$ depending on which is in $[0, 1/2]$. Then A_α and A_β are isomorphic.*

The proof of the theorem was essentially one paragraph, and it more or less stems from elementary facts from our textbook. However it may still be worth explaining why those facts are indeed facts.

The statement $|p - q| < 1$ is exercise 2.1 in the textbook, and the proof comes from the fact that using $f(p) = p - 1/2$ one gets by spectral mapping theorem (and the fact f is normal) that $|f(p)| = 1/2$ for every projection, and so $|p - q| < |f(p)| + |f(q)| = 1$. The fact that separable means countably many unitary equivalence classes stems immediately from homotopy equivalence implying unitary equivalence (and seperable means countably many homotopy classes by prop 2.2.4).

Perhaps the statement that a unique normalized trace range will be isomorphic invariant isn't immediately obvious, however it comes from the fact that there is a unique group homomorphism from the k_0 group by the trace, and under the isomorphism between k_0 groups one would get another trace, but there exists only one trace (by assumption), so the two are equivalent (and so the range of both traces must be the same).

And so theorem 2 stems almost immediately, with the final note that $\beta = \alpha \; or \; 1 - \alpha$ by noting that as both are in $[0, 1]$, then clearly α is not in $Z + Z\beta$ unless α is a multiple of β, but if the multiple isn't 1, then then two groups cannot be equal as β would not be in $Z + Z\alpha$.

4.3 Theorem 3

The statement of the theorem is:

Theorem (Theorem 3)**.** *Let α and β be irrational numbers in $[0, 1/2]$, and let m and n be positive integers. Then if $M_n \otimes A_\alpha$ is isomorphic to $M_m \otimes A_\beta$, then $m = n$, and $\alpha = \beta$.*

and the proof is rather well detailed in the paper, and I don't think there's much to add onto the technical details for it, however it's worth noting the main proposition leading to the proof is:

Proposition. *The range of the normalized trace for $M_n \otimes A_\alpha$ on projections is exactly $(n^-1(Z + Z\alpha)) \cap [0, 1]$*

which in many ways is reminiscent of theorem 1.

4.4 Theorem 4

The statement of the theorem is:

Theorem 1 (Theorem 4)**.** *The algebras A_α and A_β are strongly Morita equivalent if and only if α and β are in the same orbit of the action of $GL(2, Z)$ on irrational numbers by linear fractional transformations.*

The proof of the forward direction relies on several propositions and corollaries, but the final goal is that the ranges of the traces of A_α and A_β only differ by a scalar, and from this one gets that you can write α as a linear fractional transformation of β, and thus are in the same orbit of $GL(2, Z)$.

The other direction follows directly from noting that A_α is strong morita equivalent to $A_{\alpha^{-1}}$, and the orbit of $GL(2, Z)$ is generated by the elements that bring α to α^{-1} and $\alpha + 1$, so if α and β are in the same orbit then β is strong mortia equivalent to α^{-1} and thus α.

5 K Theory

As we almost always ask when faced with a C* algebra, at least in this class, what is the k_0 and k_1 group of A_α?

5.1 Imbeddings in AF algebras

The main theorem here is by [6] which showed that A_α can be injectively imbedded into the inductive limit of a sequence of C*-algebras, specifically:

Theorem 2 (Imbedding AF)**.** *There exists a *-monomorphism $p : A_\alpha - > A$ where A is an AF-algebra defined by an inductive limit of finite dimensional C*-algebras for which the corresponding limit of K-groups is $Z^2 - > Z^2 ...$ where each Z^2 is endowed with its natural ordering, and the connecting map phi_n is given by $phi_n = \left(\begin{smallmatrix} a_n & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where (a_n) is the continuous fraction expansion of α*

And the consequence of this is that the K_0 group of A is $Z + Z\alpha$, and thus as there exists a trace τ_1 for A such that the range is contained in $Z + Z\alpha$, and as the trace for A_α is $t = t_1 p$, we have the range of its trace is contained in $Z + Z\alpha$, which was important for the results discussed in the projections section, surveying Marc A. Rieffel's work.

The construction of the algebras in the indcutive limit are rather detailed and non-trivial, and I refer the reader to the original paper [6] for its construction and the series of lemmas and calculations that led to the above theorem.

5.2 Inductive Limit

Whereas the above work only characterized a restriction on range of the trace, and not necessarily what the k_0 group of A_α is, the work done in [2] characterizes the k_0 group explicityly, and not-surprisingly it is simply $Z + Z\alpha$ with order unit 1.

Theorem (Inductive Limit)**.** *There exists a sequence $A_1 - > A_2 ..$ of finite direct sums of matrix algebras over $C(\mathbf{T})$ with inductive limit isomorphic to A_α.*

A given sequence of such algebras has this property iff the inductive limit is simple and unital, has a unique trace and has order-unit K_0-group isomorphic to $(Z + Z\alpha, 1)$ and K_1 group isomorphic to Z^2.

the proof of this is described in the paper, but perhaps to explain why lemma 3 and theorem 1 imply that you can approximate the canonical generators arbitrarily close, note lemma 3 essentially ensures infinitely large q, q' while bounding the γ from theorem 1 (bounded by a neighbourhood from 1/4 to 4), so we see the closest elements to the generators in the approximation from theorem 1 can become arbitrarily close as they are within a distance $C(\gamma)max(1/q, 1/q')$ which can go to 0 as q, q' can go to inf while $C(\gamma)$ stays bounded.

6 Ext group

Recall first that the Calkin algebra is the quotient space of $B(H)$, i.e bounded linear operators, with $K(H)$, the compact operators. Sticking with the notation in [7] we'll define this as $L/k(H)$. Defining the $Ext_s(A_\alpha)$ and $Ext_w(A_\alpha)$ as classes of *-monomorphisms from A_α into the calkin algebra, and the homomorphism ϕ from $Ext_s(A_\alpha)$ to Z^2 as $\phi([\tau]) = (ind(\tau(u)), ind(\tau(v))$ we get the following theorem from [7]:

Theorem. *The map phi is an isomorphism. Furthermore, weak equivalence classes are strong equivalence classes, so that $Ext_s(A_\alpha)$ and $Ext_w(A_\alpha)$ coincide. The Ext group topology on them is the discrete topology.*

The study of the Ext group for C* algebras began with the work of L. G. Brown, R. G. Douglas and P. A. Fillmore in [1], and later work built on characterizing it.

A perhaps useful theorem to seeing the connection of the Ext group to K-theory is given as theorem 3.1 in [10], which is

Theorem. *Let B be any C* algebra, then there are natural isomorphisms such that $Ext(C, B) \cong K_1(B)$ and $Ext(C_0(R), B) \cong K_0(B)$*

The proof for which relies on the work done in [5].

7 Using Irrational Rotations

7.1 Flows over an Irrational Rotation

An interesting question is how to study the flows over a rotation by irrational number, and there has been a relatively recent body of literature investigating the topic [4], [3]; I sadly have no experience with the particular topic they are dealing with, but this does show that the study of irrational rotations are not limited to C* algebras.

8 Final Remarks

The point of this essay, as stated at the beginning, was to consolidate as much of the major work done on the irrational rotational algebra A_α into one document. I will have inevitably missed things, but the point was to see how the various topics in C* algebras manifest themselves in the case of A_α. As such we saw characterization of the trace, how this relates to projections, the use of inductive limits to characterize the K_0 and K_1 groups, and finally a study of the Ext groups.

Having surveyed this, a question I have, and perhaps is the topic for a future essay or work, has to do with how this relates to representations. If one looks back to the crossed product definition, the ϕ used to define A_α is simply the unitary irreducible representations of the topological group $G = (R/Z, +)$ acting on $L^2(\mathbf{T}) = L^2(R/Z)$. So could it be said that all this work has studied is a class of representations as a C* algebra, and knowing that this is in fact an irreducible representation, do analogous theorems and claims follow for general representations of G on $L^2(\mathbf{T})$?

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