Graphs and C* Algebras

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Preface

A question I had was whether there might be relations between graphs and C^*-algebras; at the time of writing this I was taking a graduate course in combinatorics, and it felt natural that combinatorial structures had algebraic analogs. That is what this essay will be about, as we investigate directed graphs and how we can derive a Cuntz-Algebra for them, and then see how the properties of this algebra are determined by the properties of the digraph, following the work done in [3].

As the focus of this paper is quite specific, I attempt to explain the prerequisite topics and proofs in more detail.

1 Directed Graphs

First let's define what we mean by a graph. Abstractly a graph is a pair of sets $G = E, V$, where V is a set of some elements (called vertices) and E is a set of 2-tuples (v_i, v_j), called edges, where $v_i, v_j \in V$. We can consider V to correspond to indices in some subset of the natural numbers, and this will be called a labelled graph; dually we can consider any vertex to be just the same as any other vertex, except perhaps for the 2-tuples they appear in E, and these we'll call unlabelled graphs. The distinction is perhaps grammatical in nature, as there is still some distinction in an unlabbeled graph to define the edges as mentioned, so it might perhaps be more intuitive to think of an unlabelled graph as an isomorphism class of labelled graphs upto permutation of indices. For the rest of the paper we'll focus on these unlabelled graphs.

There is additional structure we can impose on a graphs, which is that we can say the ordering of the vertices in the 2-tuple defining edges matter; we can say that an edge (v_i, v_j) defines an "out-edge" from v_i and an "in-edge" to v_j. Graphs with this additional structure are called directed graphs, and for these it will make sense to define them as a pair of sets and a pair of function $G' = E, V, r, s$ where E and V are as before, but $r, s : E \to V$ are the functions that define the in and out vertex of an edge (i.e define their ordering).

To help visually distinguish graphs and digraphs (shorthand for directed graphs), one might draw simple lines for the edges of a graph, and arrows for the edges of a digraph, as depicted in figure 1.

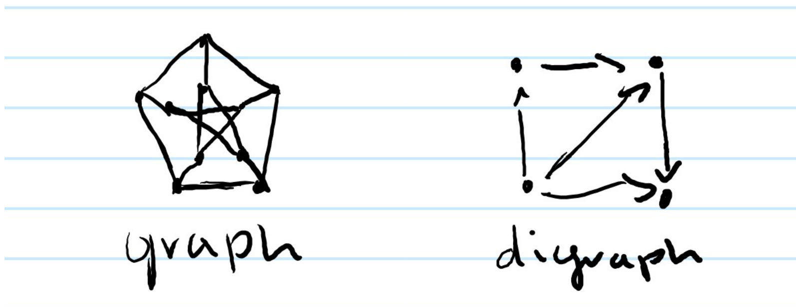

Figure 1: A regular graph and a digraph

1.1 Other structures of graphs

For a regular graph, we have a canonical adjacency matrix A, which has entries 0 or 1 depending on whether the entry in the matrix (v_i, v_j) is such that v_i, v_j share an edge; in general we could have more than one edge between two vertices, and so would have an entry greater than 1, but simple graphs, those with only one edge between any two points and which are perhaps the nicest to study, will at most have an entry of 1.

An adjacency matrix for a directed graph is similar, with the one distinction being that although the adjacency matrix for a regular graph is symmetrical (as an edge from v_i to v_j is an edge from v_j to v_i, and so both (v_i, v_j) and (v_j, v_i) have a 1) a digraph isn't, as we've made a distinction with ordering of edges. This is perhaps the best way to think of the distinction between digraphs and regular graphs in general, with graphs having an additional layer of symmetry which digraphs don't have.

There is an additional matrix we can associate with digraphs, which is the edge matrix, which is a square matrix with the same dimension as number of edges (i.e we enumerate the edges), which has value 1 for entry (e, j) if $r(e) = s(f)$ and 0 else; visually what this captures is whether two edges link up, i.e one edge leads to the next edge.

Lastly, as this is used to define the topology of a digraph in [4], there is a natural notion of a path for a graph, which is simply an ordered collection of vertices $v_1, v_2, \cdot, v_n$, which can be possibly infinite, such that $(i, i+1)$ is an edge in E. Once again for digraphs we make the additional requirement that ordering matters more, in that where as before $(i+1, i)$ would be a satisfactory edge (as we made no distinction), we require the edge between v_i and v_{i+1} to start at i and end at $i+1$. There is a specific type of path for directed graphs, called a loop, which as you can imagine is an edge that start and ends at the same vertex.

2 Cuntz-Krieger Algbera

Now let's also setup the main algebraic object we'll be interested in, which will be the cuntz-krieger algebra for matrices with entries 0 or 1; it's rather immediate to see how this is useful as adjaceny matrix for digraphs are of this form (as we'll be really just be focused on "simple" digraphs, i.e those with only one instance of a particular edge).

However the Cuntz-krieger algebra, introduced in [2] algebra is a specific extension of the more general Cuntz algebra, introduced in [1], and so we start our discussion there and build up to the former case which we're more interested in.

2.1 Cuntz-algebra

The simple definition of the Cuntz-algebra is given a hilbert space H and sequence of isometries $\{S_i\}_{i=1}^n$ that satisfy $\sum_{i=1}^{\infty} S_i S_i^* = 1$, we define the Cuntz-algebra O_n as $C*(S_1, \cdot, S_n)$; we can extend this to considering a possibly infinite sequence by requiring the sum be only less than or equal to 1'and the $C*$ algebra is define the same.

However this doesn't mean much at the surface, and for the sake intuition it is perhaps far more beneficial to talk about the algebra which this is isomorphic to, and which was used in the original paper [1] to find the various properties O_n satisfies.

At a high-level this algebra we'll construct is fundamentally no different than $C*(S_1, \cdots, S_n)$ as we'll proceed to construct essentially a span of the elements generated by these isometries, but the process in which we do so will perhaps shed slight on the sequential nature we can view this algebra; once I think we recognize that we're fundamentally dealing with a collections of sequences, it's perhaps not much of a stretch to try and make a connection with sequences in other areas of math, such as the paths in a graph, which is what we'll later do.

We thus begin by considering k-tuples with elements from $\{1, 2, \cdots, n\}$, i.e the indices of the set $\{S_i\}_{i=1}^n$, and defining the set of all such k-tuples as W_k^n, and furthermore define the set of all these sets for different values of k as $W_\infty^n = \cup_{k=1}^{\infty} W_k^n$; this is all for notation, as we can then for a given $\alpha \in W_k^n$, associate with it the product of isometries from $\{S_i\}$ defined as $S_\alpha = S_{\alpha_1} S_{\alpha_2} \cdot S_{\alpha_k}$.

Now there are a couple different things we can say about these product of isometries, in particular, as we required the sum to equal 1, we have $S_i S_j = \delta_{i,j}$, and more generally for $u, v \in W_k^n$, we have $S_u S_v = \delta_{u,v}$.

This sort of relation is very reminiscent of the relation between basis elements of matrices, and in fact if we let F_k^n be the $C*$ algebra generated by $\{S_u S_v^* : u, v \in W_k^n$, we have F_K^n is isomorphic to M_{n^k}, which is proposition 1.4 in [1] and uses the prior fact.

As we're interested in the collection of all these algebras for different k's (as we'll be interested in the $C*$-algebra generated by $\{S_i\}$), we'll define F^n as the union of all F_k^n for different k.

Now, let us consider S_1 and S_1^* to be special (for no real reason other than we can, all the holds for a general element), and let's call them V and V^-1 respectively. It's then a proposition in [1] (proposition 1.7) that if we look at the algebra generated by $\{S_i\}$ and $\{S_i^*\}$ (star algebra generated by $\{S_i\}$), defining this to be P, then for any $A \in P$ we have $A = \sum_{-N}^{-1} V^i A_i + A_0 + \sum_1^N A_i V^i$, where $A_i \in F^n$ are in fact unique (that is there is only one such set); intuitively what this means is that we can think of any element in this star algebra as being associated with a particular collection of elements in F^n, that is it is constructed by looking at the elements we generated for our algebras F_k^n for different k, which we really just constructed by looking at the sequences we can get from W_k^n, so it all boils down to the structure obtained by these sequences, which is quite appealing as this shows the underlying structure we need to reproduce to construct a specific instance of this algebra (which we'll shortly show leads to O_n).

At this point, all we need to do is define a norm and then take the completion of P to obtain a $C*$-algebra which is isomorphic to O_n; for this we simply take the sup norm of $\rho(X)$ ranging over $X \in P$ where ρ is a star-representation of P on a seperable hilbert space; lastly taking the completion with respect to this norm we get the algebra L, which by proposition 1.11 in [1] we have is isomorphic to O_n.

There are several important theorems related to O_n, in particular there is theorem 1.12 which states the specific choice of S_i doesn't matter, which intuitively makes sense as we used these to construct basis elements for an algebra isomorphic to a matrix algebra, and so as any such set with the desired property would still lead to an algebra isomorphic to the M_{n^k}, we'd get by extension that they are all isomorphic to each other). There are also other properties such as O_n being simple, amongst some other interesting properties. However what we're really interested in the construction of this C^*-algebra as we wish to somehow assocated with directed graphs a C^*-algebra, so we will move past the various other interesting things about Cuntz-algebras.

So now we understand what the Cuntz algebra fundamentally is, and so move on to what we're particular intersted in, which are Cuntz-Krieger algberas.

2.2 Cuntz-Krieger Algebra

The Cuntz-Krieger Algebra is constructed in an analagous way to the Cuntz algebra (perhaps best exemplified by the use of the same notation O_n for both), however the focus now is to construct such an algebra to be specific to a square n by n matrix A such that the entries of A are 0 or 1, and that no row/column is 0 (i.e at least one non-zero entry for each); in fact, as remarked in the paper [2], their construction does not change for general integer entries, but keeping to their original presentation let's focus with just 0 and 1 entries.

The technical way to do this is to move away from just isometries, and work with partial isometries, enforcing certain constraints that make them specific to the matrix. In particular, we consider a set of partial isometries ${S_i}_{i=1}^n$ in a Hilbert Space, satisfying the conditions

1. $P_i P_j = 0, i \neq j$

2. $Q_i = \sum_{j=1}^{n} A(i,j) P_j$

where $P_i = S_i S_i^*$ and $Q_i = S_i^* S_i$; note this later condition is what makes this specific to A. The Cuntz-Krieger algebra O_n is then simply the C^*-algebra generated by this set S_i, and to understand the structure we can once again focus on the algebras generated by specific products of this set; this analysis is quite analagous in its steps to what was illustrated above with the cuntz algebra, except with the need work with partial isometries, and so the technical details are slightly different.

In particular where before we looked at the C^*-algebra generated by products $S_u S_v^*$, we now define F_k to be the C^*-algebra generated by elements $S_u P_i S_v$ with $(u) = (v) = k$; the analogous relations here to what we had before is that $S_u S_v^* = Q_u \delta_{u,v}$, and $P_i Q_u P_j = \delta_{i,j}$, which together give us the relation we had before for just isometries. We then define F_A to be the closure of $\cup F_k$.

From this we then define P to be the star algebra generated from $\{S_i\}$ as before and for any element $X \in P$ we once again have an analogous equation to what we had before with $X = \sum_{(v)\geq 1} X_v S_v^* + X_0 + \sum_{(u)\geq 1} S_u X_u$, where $X_v, X_0, X_u \in F_A$.

At this point we once again define a norm on P by taking the sup-norm over a $*$-representation of P over some separable hilbert space, and the completion of P with respect to this norm is the Cuntz-Krieger algebra O_n.

Once again we see that the fundamental algebraic structure are these algebras F_k generated by different sequences of the partial isometries, and so with the additional characterization with 0,1 matrices, it's perhaps clear that we should be able to apply this to digraphs by looking at their adjacency matrices.

3 Cuntz-Krieger algebra for Digraphs

At this point all we really need to construct a Cuntz-Krieger algebra is a square matrix, whose rows/columns are non-zero, and has integer entries. This is incredibly general, and when looking at normal graphs it's immediately clear that if we have a complete graph, that is one such that there is a path between any two vertices v_i and v_j, then it's adjacency matrix will have at least a single 1 in each row (and thus column by symmetry) as if not then it has no edges to any other vertex, and so the graph can't be complete; we can thus clearly associate a Cunts-Krieger Algebra to complete graphs.

This isn't what was done in [3], rather they consider a slight relaxation on the requirement for the matrix to have non-zero rows and columns, and construct a completely analagous algbera to the Cuntz-Krieger algebra based on the edge matrix (which is in fact the same under some conditions on the directed graph).

The relaxation is for condition 2 discussed in the previous section, where they only require it to be true when the edge matrix A has a non-zero row at index i. There are also several other additional modification required throughout the paper as they considered more general directed graphs and built to a digraph with the desired properties; we'll skip this, and go straight to considering digraphs that have no sinks (i.e no vertices with only in-edges), are locally finite (i.e number of in and out-edges for each vertex is finite), and that each loop has an exit (that is the vertex also has other edges to different vertices).

If we let E be such a directed graph (rather than the edges, to be consistent with the notation in [3]), and A_E it's edge matrix, then we can find a set of isometries $\{S_e\}$ that satisfy the relaxed conditions, and we then associate to E the C^*-algebra $C^*(E)$ generated by the elements of $\{S_E\}$; the technical proof for this requires a bit a graph theory, a bit of representation theory, and a bit of ingenuity, and I suggest the reader read the paper for all the details, but the main theme was to first get something very close to what we want for a digraph (upto a representation) following the work for Cuntz-algebras, and then finding when this becomes exactly the canonical Cuntz-algebra.

All of this is to say that we can associate to these digraphs a Cuntz-Krieger $C*$-algebra based on their edge matrix, and now the question is, what does this tell us about the graph?

3.1 Loops and AF

The work and build up in [3] leads to the following dichotomoy summarized as corollary 3.10 in their work (note that cofinal simply means connected but in the specific sense of what paths are for directed graph):

Corollary. *Let E be a locally finite graph which has no sinks, is cofinal, and is such that every loop has an exit. Then $C^*(E)$ is simple, and*

1. *if E has no loops, then $C^*(E)$ is AF*
2. *if E has a loop, then $C(E)$ is purely infinite*

The proofs for both are quite interesting, and are definitely worth a look; I will omit them here, as they are essentially the main results of the paper and are quite involved to prove but it requires both insights from analyzing various graph constructions and algebraic tools.

4 Conclusion

Though we haven't completely explained the proof for this final corollary that links loops in digraphs and whether their associated C^*-algebra is AF or not, hopefully the discussion about the sequential nature of the Cuntz algebras illuminates why someone would have looked at them in the first place. There was also the remark at the beginning of section 4 about how we can immediately apply a Cuntz-Krieger algebra to a normal connected graph, and in fact I have yet to read a paper that investigates this, which is perhaps an interesting direction for some future work (or a summer project).

Lastly I would have liked to talk about the work done in [4] which further builds on groupoid structures and classifies the k groups for a specific $C*$-algebra they introduce for locally finite directed graph, but this would most likely add another several pages to this essay to do it any sort of justice.

References

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