Category Theory and Classification

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Preface

The main theme of this essay is the role of categories and functors in classification, and as such the essay is largely broken up into two mostly disjoint (though not completely unrelated) parts. The first part introduces categories and a particularly well-known result, the Yoneda lemma. The second half moves towards the classification functor as introduced in "Towards a Theory of Classification" [1], and presents the main theorems and examples given in the paper. The essay should be largely self-contained, modulo longer proofs where I refer the reader to other sources.

1 Category Theory

We briefly introduce categories and the different variations of them. We later will discuss the Yoneda lemma, an important way of making these abstract categories concrete, and the classification functor (which is the focus of this essay) is a categorical notion. In writing this section I found [2, 3, 4] to be a helpful resource.

1.1 The Definitions:

We begin with the definition of a category.

Definition 1 (Category $\mathbb{C}$). *A category $\mathbb{C}$ consists of a collection of "objects" $Obj_{\mathbb{C}}$, and for any two $A, B \in Obj_{\mathbb{C}}$ a collection of "arrows" (i.e maps) from A to B, $\mathbb{C}(A, B)$. The complete collection of arrows for all pairings is denoted $Arr_{\mathbb{C}}$. We require that $\mathbb{C}(A, A)$ have an identity arrow id_A, and that one can compose $f : A \to B$ and $g : B \to C$ to get $g \circ f : A \to C \in Arr_{\mathbb{C}}$.*

Lastly we require $f \circ id_A = f = id_B \circ f$ for $f : A \to B$, and that composition of arrows is associative.

Note that the terms "arrows", "maps", and "morphisms" will be used interchangeably. Furthermore, note that we used the term "collection" and not set; as one might imagine, the objects and arrows may not be a well-defined set. When in fact ths objects and arrows are sets, we call the category **small**. If only the collection $\mathbb{C}(A, B)$ is a set, we call it **locally small**. Some common examples of categories include:

1. the category of sets ($Sets$) with objects being sets and arrows being functions between sets
2. the category of open sets (for a topological space X) with arrows being inclusion
3. the category of groups, with objects being groups and arrows being group homomorphisms
4. the category of small categories, with maps given by functors (defined below) between small categories

It is also worth mentioning that to a given category $\mathbb{C}$, we have as associated **opposite category** $\mathbb{C}^{op}$ with the same objects, but arrows now given by f^{-1} where $f \in Arr_{\mathbb{C}}$; note we never enforced these arrows to be functions, hence f^{-1} giving more than one output is fine.

Categories are often quite large structures, but we can reason about how one category relates to another by the notion of a functor. This reduction is an important theme and is at center of the classification of "Nice" C^*-algebras, and the classification functor we present later.

Definition 2 (Functors)**.** *Given two categories $\mathbb{C}_1, \mathbb{C}_2$ and $f \in \mathbb{C}(A, B)$ and $g \in \mathbb{C}(B, C)$, a covariant functor F is a map between them (sending objects to objects, arrows to arrows) s.t $F(id_A) = id_{FA}$ and $F(g \circ f) = F(g) \circ F(f)$.*

Similarly, a contravariant functor is s.t $F(g \circ f) = F(f) \circ F(g)$ where now if $f \in \mathbb{C}(A, B)$ then $F(f) \in \mathbb{C}(FB, FC)$.

For example the K_0 group of a C^*-algebra can be (and is) treated as a functor from the category of C^*-algebras to the category of abelian groups. One also has the identity functor $id : \mathbb{C} \to \mathbb{C}$. As a last remark, note one can compose functors $F : \mathbb{C}_1 \to \mathbb{C}_2$ and $G : \mathbb{C}_2 \to \mathbb{C}_3$ to obtain functor $G \circ F : \mathbb{C}_1 \to \mathbb{C}_3$.

1.2 Types of Objects

We now proceed to define some important types of objects

Definition 3 (Initial and Terminal Objects)**.** *An object 0 in a category $\mathbb{C}$ is called initial if $\exists!$ map from 0 to every object in $\mathbb{C}$. Similarly and object 1 is called terminal (or final) if there $\exists!$ map from every object to 1.*

Though these objects might not always exist, if they do they are in fact unique upto isomorphism;

Lemma 1 (Uniqueness of Initial and Terminal)**.** *Initial and terminal objects are unique upto isomorphism*

Proof. Say you have two initial (or terminal) object A, B. Then by existence of maps we have $f : A \to B$ and $g : B \to A$, so $g \circ f : A \to A$. Then by uniqueness we have $g \circ f$ is the only map from A to itself, but note we always have id_A from A to itself. Thus $g \circ f = id_A$, and so we conclude they are isomorphisms. □

If an object is both the initial and terminal object, we say it is the **zero** object.

1.3 Types of Functors

We mention two properties of functors now, as we will later see the "Yoneda embedding" will satisfy these properties.

Definition 4 (Faithful and Full)**.** *A functor F from $\mathbb{C}$ to $\mathbb{D}$ is faithful if for every $f, g \in \mathbb{C}(A, B)$, $F(f) = F(g)$ iff $f = g$. We say a covariant functor F is full if F is surjective onto morphisms from $\mathbb{D}(FA, FB)$ (or $\mathbb{D}(FB, FA)$ if contravariant). If a functor is both faithful and full, we say it is fully faithful.*

2 Yoneda Lemma

The goal of this section is to present the Yoneda lemma. One can often think of this as a way of taking an abstract category into a "concrete" set category. Though we will not explicitly discuss it for the later material, it is still worth mentioning here. In writing this section I found [4] a useful resource.

2.1 The h_A and h^A functors

We first begin with a natural functor from a locally small category $\mathbb{C}$ to the category of sets, often called the "Hom functor".

Definition 5 (h_A)**.** *For a given object $A \in Obj_{\mathbb{C}}$, we associate a covariant functor from $\mathbb{C}$ to the category of set h_A defined by: takes objects X to the set $\mathbb{C}(A, X)$, takes morphisms $f : X \to Y$ to the arrow between sets $h_A(f) : \mathbb{C}(A, X) \to \mathbb{C}(A, Y)$ given by $f \circ g$ for $g \in \mathbb{C}(A, X)$.*

It's worth mentioning that an alternative notation for $\mathbb{C}(A, X)$ is $Hom(A, X)$, and hence h_A is sometimes written as $hom(A, \cdot)$ (this also explain why we use an h to denote the functor). We also have a completely analogous contravariant functor to the above.

Definition 6 (h^A)**.** *For a given object $A \in Obj_{\mathbb{C}}$, we associate a contravariant functor from $\mathbb{C}$ to the category of set h_A defined by: takes objects X to the set $\mathbb{C}(X, A)$, takes morphisms $f : X \to Y$ to the arrow between sets $h_A(f) : \mathbb{C}(X, A) \to \mathbb{C}(Y, A)$ given by $g \circ f$ for $g \in \mathbb{C}(X, A)$.*

What we have essentially established is a natural way to associate an object A to a functor from $\mathbb{C}$ to the category of sets.

2.2 Natural transformations

We need to now digress and understand what a morphism from a functor to another functor is. This is given by the following definition.

Definition 7 (Natural Transformation)**.** *Given functors F, G from $\mathbb{C}_1 \to \mathbb{C}_2$, a natural transformation $\phi : F \to G$ associates to each object $A \in Obj_{\mathbb{C}}$ a morphism $\phi_A : F(A) \to G(A)$, and is s.t given morphsim $f : A \to B$ we have $G(f) \circ \phi_A = \phi_B \circ F(f)$ as morphisms from $F(A) \to G(B)$.*

In fact given these morphisms we can define the category of functors.

Definition 8 (Category of functors)**.** *Given categories $\mathbb{C}_1, \mathbb{C}_1$, we obtain the category of functors (from $\mathbb{C}_1$ to $\mathbb{C}_2$) $[\mathbb{C}_1, \mathbb{C}_2]$ with morhpisms being the natural transformations between functors.*

A notable example of a category of functors associated to a category $\mathbb{C}$ is $[\mathbb{C}^{op}, Sets]$, called the **presheaf of** $\mathbb{C}$. As another remark, note $Nat(F, G) \coloneqq \mathbb{C}[F, G]$ where $\mathbb{C}$ is a category of functors (and $Nat(F, G)$ meaning the natural transformation from F to G), however we will stick to writing $Nat(F, G)$ as that is the common terminology.

Building on this, we say a functor $F \in [\mathbb{C}, Sets]$ is **representable** if $F \cong h_A$ for some $A \in Obj_{\mathbb{C}}$.

2.3 The Lemma

With all this said, we can finally state the lemma

Lemma 2 (Yoneda Lemma)**.** *Given a category $\mathbb{C}$ and any covariant functor $F : \mathbb{C} \to$ Sets and object $A \in Obj_{\mathbb{C}}$, we have a bijection between $Nat(h_A, F)$ $F(A)$. In fact the bijection is given by $\phi \to \phi_A(id_A)$.*

A proof is given in [4]. An alternative way of understanding this lemma is by considering the subcategory of representable functors of $[\mathbb{C}, Sets]$. Calling this subcategory R, we have $R(h_A, h_B) \cong h_B(A) = \mathbb{C}(B, A)$, a particularly nice expression relating the morphisms between objects to the natural transformations between their Hom transformations.

2.4 Some Consequences

In fact the previous remark gives us the contravariant Yoneda embedding sending objects A to $h_A \in [\mathbb{C}, Sets]$ and $f \in \mathbb{C}(B, A)$ to natural transformations in $Nat(h_A, h_B)$. Considering C^{op} we get a covariant embedding, stated below.

Definition 9 (Yoneda Embedding)**.** *We have a covariant functor called the "Yoneda Embedding" from $\mathbb{C}$ to $[\mathbb{C}^{op}, Sets]$ given by sending objects A to h^A, and $f \in \mathbb{C}(A, B)$ to natural transformations $h^A \to h^B$.*

In fact we have the Yoneda embedding is fully faithful.

Corollary 1. *The Yoneda embedding is fully faithful*

The proof is given in [4]. Note this then tells us $h_A \cong h_B$ iff $B \cong A$ by definition of fully faithful (the full gives onto all the representable functors, and faithful gives the isomorphism condition). In particular we now have a way to relate the abstract objects in a category to functors from that category to sets, in a way making the category more concrete as remarked in the opening paragraph of this section.

3 Categorical Classification

We now move towards defining categorical notions of classification, as proposed by [1]. The material presented here will be based on [1], and could be viewed as a condensed presentation of the results in that paper.

3.1 Beginnings

We begin with the main definition we study.

Definition 10 (Classifying Category and Functor)**.** *Given a category* $\mathbb{C}$*, a classifying category* $\mathbb{C}'$ *is s.t* $\exists$ *a functor* $F : \mathbb{C} \to \mathbb{C}'$ *s.t* $FA \cong FB \in \mathbb{C}'$ *iff* $A \cong B \in \mathbb{C}$*. We call this functor the classifying functor.*

As a remark, note the difference between the above definition and a faithful functor is that in the above we are looking at objects, where as with a faithful functor we are looking at morphisms.

The goal is to ask when such functors exists, and what the classifying category would be (which are preferably "simpler categories"). Note if all objects were isomorphic in $\mathbb{C}$, then quotienting out isomorphism gives a classifying category with the classifying functor being the quotient map; in this way our main issues to existence have to deal with homomorphisms.

A first approach is to simplify the problem by quotienting out the homomorphisms between objects differing by an automorphism in the domain or codomain (or both); would this give us a classifying category?

Lemma 3. *Quotienting the homomorphisms differing by an automorphism does not always give a category*

Proof. Proof by counter-example, where we see the composition of two maps does not belong to a single equivalence class contradicting well-defined. Consider the category *Sets*. Note the composition of two non-constant maps can be constant; one trivially obtain such a maps by taking the second map to be s.t the image of the first gets sent to a single element. However the product of the equivalence classes of these maps always contains a non-constant map (by shuffling the image of the first to be in the preimage of two different values for the second map). However constant maps are equivalent only to constant maps (when composing or precomposing by automorphism). Thus we have the product of the equivalence classes of the two maps gives a map belonging to two equivalence classes. This gives a contradiction to well-definedness. □

3.2 The First Theorem

In light of this we shall consider categories with a notion of "inner automorphism", i.e conjugation. Examples of such categories are groups (or all categories with a group action). The important fact about inner automorphisms is by the definition of a morphism on groups, they factor through: $f(g^{-1}ag) = f(g)^{-1}f(a)f(g)$. This property gives the categorical axiom we need.

Definition 11 (Inner-automorphism Axiom)**.** $g \in \mathbb{C}(A, A)$ *is an inner-automorphsim, then for all* $f \in \mathbb{C}(A, B)$*,* $f \circ g = h \circ f$ *where* $h \in \mathbb{C}(B, B)$ *is an inner-automorphism.*

Any collection of automorphism associated to each object which are consistent with the axiom is sufficent for the following result, and we will call them inner-automorphisms. Do note then the inner-automorphism for a given object are in fact a subgroup under composition (which one might call a normal subgroup in the group-theoretic setting).

Theorem 1. *Let* $\mathbb{C}$ *be a category with an inner-autormorphsim (given by the above axiom). Then* $\mathbb{C}^{out}$ *obtained by quotienting out the inner-automorphisms, is a category and moreover a classifying category of* $\mathbb{C}$ *with the the quotient map functor.*

I refer the reader to [1] for more explanation on the historical context of the result. All that is needed to prove the result is to show $\mathbb{C}^{out}$ is a category, and this follows by looking at what went wrong in the case of automorhisms (stated as a lemma earlier). As a last remark, in practice this is applied to categories with a group structure, so one can ignore the abstraction of inner-automorphisms and consider the concrete action one is used to.

3.3 The Second Theorem

The above result, while interesting, can be made stronger if there is a notion of topology over the morphisms of $\mathbb{C}$, $Arr_{\mathbb{C}}$.

In particular say each $\mathbb{C}(A, B)$ has a complete metric structure; We would like it satisfy two additional "compatibility" conditions:

1. For any objects A, B, C the map $\mathbb{C}(A, B) \times \mathbb{C}(B, C) \to \mathbb{A}, \mathbb{C}$ is jointly continuous in both $\mathbb{C}(A, B)$ and $\mathbb{C}(B, C)$.

2. Composition by an inner-automorphism is an isometry, i.e for any f an inner-isomotry on $\mathbb{C}(B, B)$ for arbitrary B, then $\mathbb{C}(X, B) \to \mathbb{C}(X, B)$ given by $f \circ g$, is an isometry

With these conditions one has the following theorem.

Theorem 2. *Let* $\mathbb{C}$ *be a category with an inner-automorphism (as stated earlier) and* $Arr_{\mathbb{C}}$ *has a complete metric structure satisfying the previous two axioms. We then have* $\overline{\mathbb{C}^{out}}$ *is a category (with same objects but morphism now* *the closure of equivalence classes of morphisms by inner-automorphisms), and moreover the quotient map is a classifying functor. One could call this a strong classification as isomorphisms in* $\overline{\mathbb{C}^{out}}$ *lift to* $\mathbb{C}$.

The proof uses the intertwining argument found in much of the classification of C^*-algebra literature, and the explicit proof is given in [1]. Note the main addition given with this theorem is that the classification category contains slightly more morphisms (the closure) but in return also gives a lifting property.

3.4 Applying the Second Theorem

The last question at hand is when do categories satisfying the axioms needed for Theorem 2 exist? In what follows we briefly describe some examples representative of the types of examples described in [1]. The theme is taking advantage of a countable structure, and defining a metric according to the discrete structure (or existing topology) of the objects.

Countable Groups In this category $\mathbb{C}$, inner-automorphism is given by the usual group definition of conjugation. As objects G, H are countable, we can enumerate them and define the following complete metric on $f, h \in \mathbb{C}(G, H)$:

$$d(f,h) = \sum_{g \in G} 2^{-n} \delta_{f(g),h(g)}$$

One can check that the triangle inequality is satisfied (follows from the kronecker delta satisfying triangle inequality) and is positive definite (follows from summing over all elements).

Furthermore, one can check the two topological compatability of Theorem 2 are satisfied: the proofs for both amount to writing out the definitions and rearranging (see [1] for the second).

Thus we have an example of a category satisfying Theorem 2, where the topology on morphisms is given by a type of discrete point-wise topology.

Seperable C^*-algebras In this category $\mathbb{C}$, the inner-automorphisms are given by conjugation with unitary elements (where now note inverse is the same as taking the adjoint). Note that being seperable (for a complex-algebra) is equivalent to having a sequence $(a_n)_{n \in \mathbb{N}}$ s.t the span of the sequence is all of the algebra A. Then we define the following complete metric on $\mathbb{C}(A, B)$ by:

$$d(f,h) = \sum_{n \in \mathbb{N}} 2^{-n} ||f(a_n) - h(a_n)||$$

Note the homomorphism are linear, hence agreeing on the generating set mean they agree everywhere, showing this is in fact positive definite. We also have the two conditions for Theorem 2 are satisfied: the first follows similarly to the second where one can use that finite sets of points form a compact subset, and the second follows from conjugation by unitaries being an isometry.

Thus we have another example of a category satisfying Theorem 2.

References

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