The Classification of "Nice" C^*-algebras

Anvith Thudi

Preface

The goal of this essay is to build up towards the classification of "sufficiently nice" C^*-algebras, focusing on covering all the topics needed to understand the final statement. We will omit proofs, focusing more on the exposition of topics and making it as self-sufficient as possible. Much of the material, in particular definitions and theorems, will be based on Karen Strung's "An Introduction to C^*-algebras and the Classification Program" [1].

C^*-algebra Definitions

Introductory Definitions A C^*-algebra is a complete normed $\mathbb{C}$-algebra with an involution * satisfying $(a^*)^* = a$,$(ab)^* = b^*a^*$ and $||a^*a|| = ||a||^2$. Notably this is a Banach algebra, and hence the theory and definitions of Banach algebras can be applied. As such a *seperable* C^*-algebra is one that has a countable dense subset, or equivalently, a countable number of generators (note X is a set of generators for the C^*-algebra A if $C^*(X) = A$); a *unital* C^*-algebra is one with a multiplicative identity; a *simple* C^*-algebra is one with no nontrivial ideals.

Some specific elements of interest are *projections* which satisfy $a^2 = a$ and $a^* = a$, *unitaries* which satisfy $a^* = a^{-1}$, *isometries* which satisfy only $a^*a = \mathbf{1}$ (note unitaries are isometries, but not vice-versa), and *partial isometries* which satisfy $v = vv^*v$. If an element satisfies $a^* = a$ we call it *self-adjoint*.

We also associate to every $a \in A$ the *spectrum* $Sp(a) = \{\lambda \in \mathbb{C} : (a - \lambda\mathbf{1})$ not invertible$\}$; it follows that an element is self-adjoint iff $Sp(a) \subset \mathbb{R}$, and we say an element is *positive*, i.e $a > 0$, if $Sp(a) \subset \mathbb{R}^+$.

There are also notions of equivalences between elements. Two projections p, q are Murray Von-Neumann equivalent if there is a partial isometry v such that $p = v^*v$ and $q = vv^*$. We also have the usual homotopy equivalence of elements (induced by the norm topology).

Nuclear C^*-algebras We will define *Nuclear* C^*-algebras not by the typical definition with tensor norms, but by satisfying the *completely positive approximate property.* This approach will be relevant for our later notion of nuclear dimension.

First recall a map $\theta : A \to B$ is called completely positive if the extension $\theta^n : M_n(A) \to M_n(B)$ by applying θ entry-wise is positive $\forall n \in \mathbb{N}$. Now given finite $\mathfrak{F} \subset A$ and $\epsilon > 0$, we say $(\mathfrak{F}, \epsilon)$ is a *completely positive approximation* for A if $\exists$ (F, ψ, ϕ) where F is a finite dimension C^*-algebra, $\psi : F \to A$ and $\phi : A \to F$ are completely positive maps, and $||\phi \circ \psi(a) - a|| < \epsilon$ $\forall a \in \mathfrak{F}$.

We then say A has the completely positive approximate property if for every finite $\mathfrak{F} \subset A$, $\epsilon > 0$, there is a $(\mathfrak{F}, \epsilon)$ completely positive approximation with ψ, ϕ both contractions (i.e operator norm < 1). It has been shown that in fact A is nuclear iff it satisfies the completely positive approximate property [2, 3].

$\mathbf{K_0}$ and $\mathbf{K_1}$

We have two important abelian groups associated to a C^*-algebra A, the first we discuss is the K_0 group and then the K_1 group. Together they make up what is often called the K-theory of a C^*-algebra.

$\mathbf{K_0}$ Let $M_n(A)$ be matrices with elements of A (which is itself a C^*-algebra) and note we can say $M_n(A) \subset M_m(A)$ where $m > n$ by including matrices in $M_n(A)$ to the top left corner of matrices of dimension m. Thus the set $M_\infty = \cup_{n \in \mathbb{N}} M_n(A)$ makes sense as an increasing sequence of subsets. Moreover we have any $p, q \in M_\infty$ belong to some $p \in M_n(A),\ q \in M_m(A)$ and hence can define the operation $p \bigoplus q \in M_{n+m}(A)$ by placing p, q along the diagonal (in that order).

We can define an equivalence on the projections in M_∞, analogous to Murray Von-Neumann equivalence, by setting $p \sim q$ if $\exists v \in M_{m,n}$ such that $v^*v = p$ and $vv^* = q$. Notably we have the following properties

1. if $p_1 \sim p_2$ and $q_1 \sim q_2$ then $p_1 \bigoplus q_1 \sim p_2 \bigoplus q_2$
2. $p \bigoplus q \sim q \bigoplus p$
3. if $p, q \in M_n(A)$ and $pq = 0$ then $p + q\ p \bigoplus q$
4. $(p \bigoplus q) \bigoplus r\ p \bigoplus (q \bigoplus r)$

These properties imply that $M_\infty / \sim$ is an abelian semi-group under the operation $\bigoplus$ (where I abuse notation here as by M_∞ I am only referring to the projections), and thus we can define K_0 as the Grothendick completion of this semi-group, i.e $K_0(A) = \{[p] - [q] : [p], [q] \in M_\infty(A)/ \sim\}/ \sim_G$ where $[p] - [q] \sim_G [p'] - [q']$ iff $[p] + [q'] + [r] = [p'] + [q] + [r]$ for some $[r]$. Note that this construction, in particular the use of a the grothendick construction, bears significant similarity to the topological K-group construction I described in my previous essay.

Ordered Group Structure with $\mathbf{K_0(A)_+}$ Recall that an ordered abelian group G is a group with a partial order that also satisfies $x < y$ implies $x + z < y + z$ for all $z \in G$, and $G = \{x \in G : 0 \leq x\} - \{x \in G : 0 \leq x\}$ (i.e decomposes into positive and negative elements). Letting G_+ denote the positive elements, we then usually denote the ordered group by (G, G_+).

Note that $K_0(A)$ is constructed as the difference of projections $[p]$, hence we can define $K_0(A)_+ = \{[p] : p \in M_\infty(A)\ p\ \textit{is a projection}\ \}$ and note then $(K_0(A), K_0(A)_+)$ gives us a partially ordered group with $x < y$ if $y - x \in K_0(A)_+$.

As A is unital we have a unit $\mathbf{1}$, and in fact $[\mathbf{1}]$ is an *order unit* (i.e $\forall x \in K_0(A)$ have $n \in K_0(A)$ s.t $-n\mathbf{1} \leq x \leq n\mathbf{1}$). So $(K_0(A), K_0(A)_+, [\mathbf{1}])$ is an ordered abelian group with distinguished order unit, or a *pointed ordered abelian group.*

Formally we define this ordered group structure only for unital stably finite C^*-algebras (i.e $M_n(A)$ is finite for every $n \in \mathbb{N}$). However, it still makes sense to refer to $K_0(A)_+$ (given by the prior definition) and $[\mathbf{1}]$ for non stably-finite C^*-algebras, and this will be a part of our later classification invariant.

$\mathbf{K_1}$ In the case of a unital C^*-algebra A we can also associate another group K_1. Let $U_n(A)$ be the unitaries of $M_n(A)$, and thus analogous to before $U_\infty = \cup_{n\in\mathbb{N}} U_n$ makes sense, and we have an operation $\bigoplus$. In this case we now take $u \sim v$ for $v \in U_n(A),\ v \in U_m(A)$ if $\exists k \geq max\{n, m\}$ s.t $u \bigoplus \mathbf{1}_{k-n}$ is homotopy equivalent to $v \bigoplus \mathbf{1}_{k-m}$ in the $U_k(A)$ subspace topology of $M_k(A)$.

It can be shown that $U_\infty(A)/\sim$ is an abelian group with the operation $\bigoplus$, and hence we define $K_1(A) = U_\infty(A)/\sim$

The Invariant

With that we can now give what will eventually be our invariant used in the classification of "nice" C^*-algebras, called the *Elliott Invariant.*

Tracial States Recall a state on a C^*-algebra is a linear functional $\phi : A \to \mathbb{C}$ s.t $||\phi|| = 1$. Tracial states are states s.t we also have $\phi(ab) = \phi(ba)$, and we denote the set of all tracial states by $T(A)$.

State Space of Ordered Groups First we define the state space of an ordered group with distinguished unit (G, G_+, u) as the set of group homomorphism $\phi : G \to \mathbb{R}$ s.t $\phi(G_+) \subset \mathbb{R}^+$ and $\phi(u) = 1$. Note this is the natural extension of the typical definition of positive states on a Banach algebra, and we denote it by $S(G)$.

The Elliott Invariant With that we then have the 6-tuple associated to any simple, seperable, nuclear, unital C^*-algebra A given by $Ell(A) = ((K_0(A), K_0(A)_+, [\mathbf{1}]), K_1(A), T(A), \rho_A)$ where $\rho_A : T(A) \to S(K_0(A))$ is given by $\rho(\tau)([p] - [q]) = \tau(p) - \tau(q)$. Though it would be nice for this to classify all simple,seperable, nuclear, unital C^*-algebras, it is not.

Jiang-Su Algebras

The main counter-example to the Elliott invariant classifying all simple,seperable, nuclear, unital C^*-algebras A is that exists C^*-algebra Z s.t for any A $Ell(A \bigotimes Z) = Ell(A)$ but there are cases where $A \ncong A \bigotimes Z$. We won't get into the proofs of those statements, however we will define this C^*-algebra Z, called the *Jiang-Su algebra*; later we will see being Z-stable, which turns out to be equivalent to having finite nuclear dimension (which we will later define), will be one of our niceties.

Inductive Limit of C^*-algebras We first will discuss a method of construction of C^*-algebras as a limit of other C^*-algebras.

An *inductive sequence* of C^*-algebras $(A_n, \phi_n)_{n\in\mathbb{N}}$ is a sequence of C^*-algebras A_n with *-homomorphisms $\phi_n : A_n \to A_{n+1}$. For such an inductive sequence we can define $\mathfrak{A} = \{(a_j)_{j\in\mathbb{N}} \in \prod A_n : \exists N \; s.t \; a_{j+1} = \phi_j(a_j) \; \forall j \geq N\}$, and note it is a *-algebra under point-wise operations, and in fact $\rho(a) = \lim ||a_j||_{A_j}$ is a C^*-seminorm (recall a C^*-seminorm is just a C^*-norm but is only positive and not necessarily positive definite).

If we now take the ideal $N = ker(\rho)$ and consider $A = \overline{\mathfrak{A}/N}$ we get a C^*-algebra, which is the enveloping algebra of $(\mathfrak{A}, \rho)$. We call A the *inductive limit.*

It's worth noting we can make sense of $\phi_{n,m} : A_n \to A_m$ by $\phi_m\phi_{m-1}\cdots\phi_n$, and also $\phi^n : A_n \to A$ by $\phi^n(a) = (a_j)_{j\in\mathbb{N}}$ where $a_j = 0$ for $j < n$, $a_n = a$, $a_j = \phi_{n,j}(a)$ for $j > n$. We remark on this as there is a universal property associated to inductive limits, which is that if there is a C^*-algebra B and associated *-homomorphisms $\psi^n : A_n \to B$ s.t $\psi^n = \psi^{n+1} \circ \phi_n$, then there is a unique *-homomorphism $\psi : A \to B$ s.t $\psi^n = \psi \circ \phi^n$.

Dimension-Drop Algebras Let $p, q, d \in \mathbb{N} \setminus \{0\}$ with both p, q divisors of d, then we define the *dimension-drop algebra* $I(p, d, q) = \{f \in C([0, 1], M_d(\mathbb{C})) : f(0) \in M_p \bigotimes 1_{d/p} \text{ and } f(1) \in M_q \bigotimes 1_{d/q}\}$. Note if p, q are relatively prime, i.e pq is the least common multiple, then $I(p, pq, q)$ is called a *prime dimension-drop algebra.*

Dimension Drop Algebras satisfy several important properties.

Proposition 1. *$I(p, d, q)$ has no nontrivial projections iff p, q are relatively prime*

and notably we can classify their K-theory

Proposition 2. *Let r be the greatest common divisor of p, q and $n = dr/(pq)$, then $K_0(I(p, d, q)) = \mathbb{Z}$ and $K_1(I(p, d, q)) = \mathbb{Z}/n\mathbb{Z}$. Notably, if $I(p, d, q)$ is a prime dimension-drop algebra, then $K_1(I(p, d, q)) = 0$.*

Jiang-Su Algebra The Jiang-Su algebra Z is constructed as the inductive limit of a special inductive sequence of prime dimension-drop algebras given by the following proposition.

Proposition 3. *There exists an inductive sequence $(A_n, \phi_n)_{n\in\mathbb{N}}$ of prime dimension-drop algebras $I(p_n, d_n, q_n)$ s.t $\phi_{m,n} : A_m \to A_n$ are injective and given by*

$$\phi^{m,n}(f) = \mathbf{u}^* \begin{pmatrix} f\circ\varepsilon_1^{m,n} & 0 & \cdots & 0 \\ 0 & f\circ\varepsilon_1^{m,n} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f\circ\varepsilon_k^{m,n} \end{pmatrix} \mathbf{u}$$

where $k = d_m/d_n$, $\mathbf{u}$ is a continuous path of unitaries in $M_{p_n q_n}$ and ε_i are continuos paths satisfying $|\varepsilon_i^{m,n}(x) - \varepsilon_i^{m,n}| < 1/2^{n-m}\ \forall x, y \in [0,1]$

Letting Z be the inductive limit of this sequence, it is shown that it is infinite-dimensional, simple, unital, nuclear, and has unique tracial state. Moreover we have $(K_0(Z), K_0(Z)_+, [\mathbf{1}_Z]) \cong (\mathbb{Z}, \mathbb{Z}_+, 1)$ and $K_1(Z) \cong K_1(\mathbb{C}) = 0$, i.e the $k-theory$ is isomorphic to that of $\mathbb{C}$. In fact, it is the **unique** such algebra satisfying all these properties.

Nuclear Dimension

Earlier we defined nuclear C^*-algebras in terms of the completely positive approximate property. We will follow a similar idea to define nuclear dimension and then decomposition rank; the importance of nuclear dimension is it being finite turns out to be equivalent to $A \cong A \bigotimes Z$, i.e A being Z-stable.

Definition First we say a completely positive contractive map $\phi : A \to B$ is *order zero* if for any $a, b \in A_+$ with $ab = ba = 0$ then $\phi(a)\phi(b) = 0$.

With that definition, if A is a seperable C^* algebra such that d is the smallest integer such that for any finite set $\mathfrak{F} \subset A$ and $\epsilon > 0$ we have a C^*-algebra F with $d+1$ ideals, $F = F_0 \bigoplus F_1 \cdots \bigoplus F_d$, and $\psi : A \to F$ contractive and $\phi : F \to A$ s.t $\phi|_{F_n}$ are all completely positive contractive order zero maps, satisfying $|\phi\circ\psi(a) - a|| < \epsilon\ \forall a \in \mathfrak{F}$, then A has *nuclear dimension* $dim_{nuc}A = d$. If also ϕ is contractive, then we say A has *decomposition rank* $drA = d$. Note if these is no such integer for any of these to hold, then we say the dimension/rank is ∞.

The Toms-Winter Conjecture For the time being let us denote $\mathfrak{E}$ as the set of all simple, seperable, unital, nuclear, infinite-dimensional C^*-algebras. Recall that one of the main issue with using the Elliot invariant to classify the set $\mathfrak{E}$ is that tensoring with Z doesn't change the invariant. As such we'd like to consider Z-stable C^*-algebras, and the Toms-Winter conjecture offers an equivalent characterization of such algebras and their possible classification.

Conjecture 1 (Toms-Winter II [4])**.** *For $A \in \mathfrak{E}$ the following are equivalent*

1. $A \cong A \bigotimes Z$

2. $dim_{nuc} A < \infty$

3. A has strict comparison of positive elements

Moreover, the set of algebras satisfying these conditions form the largest class of C^*-algebras for which Ell is a complete invariant.

It was in fact later showed that for the case of $\delta_e T(A)$ (the extreme boundary of the tracial states) being finite dimensional the three statements are equivalent. In the not restricted case, the first two are equivalent [4, 5]. So we see having finite nuclear dimension will be important to using the Elliott invariant for classification.

UCT

We now take a brief detour into one the assumptions needed for the classification, which is that the algebra satisfies the *Universal Coefficient Theorem* (UCT), which is a property coming from KK theory. It should be noted that it is still open whether this immediately comes from being nuclear, but as it is not proven, one assumes this as an additional property.

Notable Consequences To define the universal coefficient theorem requires defining KK-theory amongst many other structures, so instead we list two of the main consequences relevant for the K-theory of C^*-algebras and the goal of classification.

The first has to do with the the K-theory when tensoring algebras.

Theorem 1. *Suppose A and B are nuclear C^*-algebras and that A satisfies the UCT. If $K_0(A)$ and $K_1(A)$ or $K_0(B)$ and $K_1(B)$ are torsion free (i.e not finite-order elements), then $K_0(A \bigotimes B) \cong (K_0(A) \bigotimes K_0(B)) \bigoplus (K_1(A) \bigotimes K_1(B))$ and $K_1(A \bigotimes B) \cong (K_0(A) \bigotimes K_1(B)) \bigoplus (K_1(A) \bigotimes K_0(B))$. Note here tensoring is understood as tensoring of abelian groups as identified as $\mathbb{Z} - modules$*

The second has to do with properties of tracial states. Recall a faithful tracial state is a tracial state which satifies $\phi(a^*a) = 0$ means $\phi(a) = 0$. Also note a tracial state τ is quasidiagonal if for every finite subset $\mathfrak{F} \subset A$ and $\epsilon > 0$ there exists a unital completely positive map $\psi : A \to M_n$ to some matrix algebra M_n such that $||\psi(ab) - \psi(a)\psi(b)|| < \epsilon$ for every $a, b \in \mathfrak{F}$ and $|tr_{M_n} \circ \psi(a) - \tau(a)| < \epsilon \ \forall a \in \mathfrak{F}$. With that we also have the following.

Theorem 2 (Tikuisis-White-Winter [6])**.** *Let A be a seperable nuclear C^*-algebra which satisfies the UCT, then every faithful tracial state is quasidiagonal.*

There is also a notion of a C^*-algebra being quasidiagonal which comes from the notion of bounded operators over a Hilbert space being quasidiagonal, and hence we say A is quasidiagonal if $\exists$ a faithful representation π to bounded operators over a hilbert space s.t $\pi(A)$ is quasidiagonal. We then also have the following corollary.

Corollary 1. *If $A \in \mathfrak{E}$ and $dim_{nuc}A < \infty$ and A satisfies the UCT, then A is quasidiagonal iff $drA < \infty$.*

Generalized Tracial Rank and Classification

We now can consider the first step towards classification, which will be done by introducing generalized tracial rank and considering tensoring by the universal UHF algebra

The Universal UHF Algebra Let $(n_i) \in \mathbb{N}$ be s.t n_i divides n_{i+1} and $\phi_i : M_{n_i} \to M_{n_{i+1}}$ be unital *-homomorphisms. Then the inductive limit of the inductive system (M_{n_i}, ϕ_i) is called a UHF algebra.

Notably, to each UHF algebra we can associate a supernatural number by $\mathfrak{p} = \prod_{p\ prime} p^{k_p}$ where $k_p = \sup\{k \in \mathbb{N} : \exists m\ s.t\ p^k\ divides\ n_m n_{m-1} \cdots n_1\}$. In fact we have the following classification.

Theorem 3 (UHF algebra classification)**.** *Let U_1, U_2 bet two UHF algebras with associated supernatural number $\mathfrak{p}, \mathfrak{q}$. Then $U_1 \cong U_2$ implies $\mathfrak{p} = \mathfrak{q}$.*

The *universal UHF algebra Q* is the unique (by the above classification) UHF algebra with associated supernatural number $\prod_{p\ prime} p^{\infty}$

Generalized Trace Rank We first define the *Elliott-Thomsen building block* given by the following. Let F_1, F_2 be two finite-dimensional C^*-algebras and suppose there are two unital *-homomorphisms $\phi_0, \phi_1 : F_1 \to F_2$, and take $A = \{(f,g) \in C([0,1], F_2) \bigoplus F_1 : f(0) = \phi_0(g), f(1) = \phi_1(g)\}$; we call C^* algebras of the form of A Elliot-Thomsen building blocks.

Now if A is simple and unital, and for any $\epsilon > 0$, $c \in A_+$, finite subset $\mathfrak{F} \subset A$ there exists non-zero projection p and C^*-subalgebra which is also an Elliott-Thomsen building block B with $\mathbf{1}_B = p$ satisfying

1. $||pa - ap|| < \epsilon\ \forall a \in \mathfrak{F}$
2. $dist(pap, B) < \epsilon\ \forall a \in \mathfrak{F}$
3. $1_A - p$ is murray-von Neumann equivalent to a projection in $\overline{cAc}$

then we say A has *generalized tracial rank at most one*

Classification With that we then have the following theorem classifying certain C^*-algebras with the restriction that $A \bigotimes Q$ has generalized trace rank at most 1.

Theorem 4 ([7])**.** *Let A, B be simple, seperable, unital, nuclear, C^*-algebras, and suppose $A \bigotimes Q$ and $B \bigotimes Q$ have generalized tracial rank at most 1. Then $A \bigotimes Z \cong B \bigotimes Z$ iff $Ell(A \bigotimes Z) \cong Ell(B \bigotimes Z)$. Moreover any isomorphism between $Ell(A \bigotimes Z)$ and $Ell(B \bigotimes Z)$ can be lifted to an isomorphism of $A \bigotimes Z$ and $B \bigotimes Z$.*

The classification

Note that the previous theorem, when restricted just to $dim_{nuc}A < \infty$ so that $A \cong A \bigotimes Z$, gives us that the Elliott invariant classifies the algebras with the additional condition that $A \bigotimes Q$ has generalized trace rank at most 1. However, it was shown that this last condition is redundant when $drA < \infty$ and A satisfies the UCT.

Theorem 5 ([8])**.** *Let A be a simple, seperable, unital C^*-algebra with $drA < \infty$ and satisfies the UCT. If all the tracial states of A are quasidiagonal, then $A \bigotimes Q$ has generalized tracial rank at most one.*

But there are a couple things to note. Firstly the condition on the tracial states is redunant as for such algebras it is known that all tracial states are quasidiagonal. Also as $drA < \infty$ implies $d_{nuc}A < \infty$ we see this with the previous theorem in fact classifies all simple, seperable, unital C^*-algebras with finite decomposition rank and satisfy the UCT.

However we can further relax decompisition rank being finite to nuclear dimension being finite. For simple, seperable, unital C^*-algebras s.t $dim_{nuc}A < \infty$ and A satisfies the UCT, it is know that $drA < \infty$. Thus we equivalently have the complete classification of all simple,seperable, unital, infinite-dimensional C^*-algebras with finite nuclear dimension and which satisfy the UCT.

Theorem 6. *Let A, B be simple, seperable, unital, infinite-dimensional C^*-algebras with finite nuclear dimension and which satisfy the UCT. If there is an isomorphism $\psi : Ell(A) \to Ell(B)$ then there is a *-isomorphism $\Phi : A \to B$ which is unique upto approximate unitary equivalence, and satisfies $Ell(\Phi) = \psi$*

References

[1] K. R. Strung *et al.*, *An Introduction to C*-algebras and the Classification Program.* Springer, 2021.

[2] M.-D. Choi and E. G. Effros, "Nuclear c*-algebras and the approximation property," *American Journal of Mathematics*, vol. 100, no. 1, pp. 61–79, 1978.

[3] E. Kirchberg, "C*-nuclearity implies cpap," *Mathematische Nachrichten*, vol. 76, no. 1, pp. 203–212, 1977.

[4] W. Winter and J. Zacharias, "The nuclear dimension of c-algebras," *Advances in Mathematics*, vol. 224, no. 2, pp. 461–498, 2010.

[5] J. Castillejos, S. Evington, A. Tikuisis, S. White, and W. Winter, "Nuclear dimension of simple c-algebras," *Inventiones mathematicae*, vol. 224, no. 1, pp. 245–290, 2021.

[6] A. Tikuisis, S. White, and W. Winter, "Quasidiagonality of nuclear c*-algebras," *Annals of Mathematics*, pp. 229–284, 2017.

[7] G. Gong, H. Lin, and Z. Niu, "Classification of finite simple amenable z-stable c^*-algebras," *arXiv preprint arXiv:1501.00135*, 2014.

[8] G. A. Elliott and Z. Niu, "On the classification of simple amenable c*-algebras with finite decomposition rank," *Operator Algebras and their Applications: A Tribute to Richard V. Kadison", Contemporary Mathematics*, vol. 671, pp. 117–125, 2016.